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STABILITY ANALYSIS OF BARS WITH VARYING CROSS-SECTION

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Abstract— The exact solutions for stability analysis of bars with varying cross sections subjected to simple or complicated loads, including concentrated and variably distributed axial loads are presented in this paper. The distribution of flexural stiffness of the bar and that of axial loads acting on the bar are expressed as power functions or exponential functions; also, the extracted exact solutions are expressed in terms of Bessel functions and super geometric series.

INTRODUCTION

As is well known, the model for stability analysis of high-rise structures, tall buildings and trusses can be treated as a uniform bar or a non-uniform bar subjected to concentrated and variably distributed axial loads. The exact solutions of such bars have not previously been proposed in the stability analysis literature. The simple cases, such as a bar of varying cross sections subjected to concentrated axial loads at its top or ends, a uniform bar subjected to uniformly distributed axial loads, or a cuneiform bar carrying its own weight, were studied by Timoshenko (1930), Genik (1950), Jasinsky (1902), Karman and Biot (1940). The more complicated cases, such as buckling of columns under variably distributed axial loads, were discussed by Vaziri and Xie (1992), Arbabi and Li (1991). A new numerical model (Vaziri and Xie, 1992) and new method of transforming an eigenvalue problem in a finite dimensional subspace (Arbari and Li, 1991) were proposed by these researches. In this paper, the exact solutions for stability analysis of bars with varying cross sections subjected to simple or complicated loads, including concentrated and variably distributed axial loads, are found by selecting the suitable expressions, such as power functions and exponential functions, for the distribution of flexural stiffness of the bar and for axial loads acting on the bar. All of the exact solutions are expressed in terms of Bessel functions and super geometric series. As the selected expressions are suitable for describing the distribution of flexural stiffness and axial loads of a large number of engineering structures, especially, high-rise structures and tall buildings (Li Guiqing, 1985), the proposed method has practical significance for stability analysis.

GENERAL DIFFERENTIAL EQUATION OF BENDING AXIS

When the axial force acting on a straight bar reaches its critical value, the straight equilibrium form will not be stable, but equilibrium is possible. The general differential equation of bending of a bar is derived as follows.

Consider the element dx at the position x. Since the element is in equilibrium and the deformation state of the bar is small, the following equation can be derived from

$$\Sigma F_y = 0 \quad \text{(Fig. 1)}$$

as

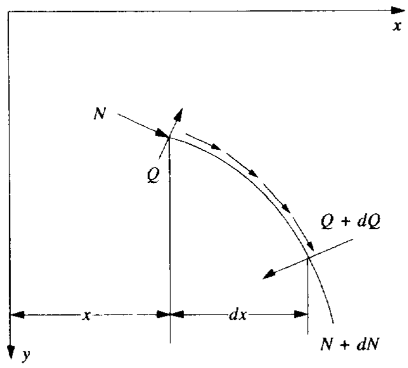

Fig. 1. Model.

$$Ny' - (N + \mathrm{d}N)(y' + \mathrm{d}y') - Q + (Q + \mathrm{d}Q) = 0.$$

Neglecting infinitesimal quantities of the second order and dividing every term by dx, one obtains

$$(Ny')' - Q' = 0. \tag{1}$$

Similarly, from $\Sigma F_x = 0$ it is found that

$$N' + (Qy')' = -q. \tag{2}$$

Using the equation of moment of the force acting on the element, dx, about one point one may derive

$$M' = Q. \tag{3}$$

Substitution of eqn (3) into eqn (1) gives

$$(Ny')' - M'' = 0. \tag{4}$$

The integral of eqn (4) results in

$$Ny' - M' = C_0. \tag{5}$$

Dividing every term of eqn (5) by N we have

$$y' - \frac{M'}{N} = \frac{C_0}{N}. \tag{6}$$

Substituting the moment–curvature relation in the derivative of eqn (6) one yields

$$\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} - \frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}x}\frac{\mathrm{d}M}{\mathrm{d}x} + \frac{N}{EJ}M = \frac{C_0}{N}\frac{\mathrm{d}N}{\mathrm{d}x}. \tag{7}$$

It can be seen from eqn (5) that $C_0 = -M' = -Q$ when $y' = 0$. This fact means that C_0 is equal to the absolute value of shear force at the section, but it has a contrary sign.

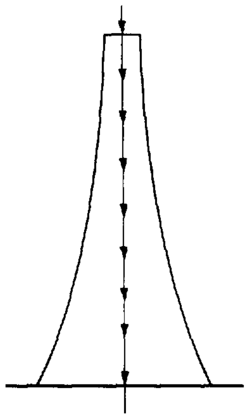

Fig. 2. General cases.

STABILITY ANALYSIS OF NON-UNIFORM CANTILEVERS SUBJECTED TO CONCENTRATED AND VARIABLY DISTRIBUTED AXIAL LOADS

It is evident that $C_0 = 0$ for cantilevers with the origin at free end (Fig. 2). In this case, eqn (7) becomes

$$\frac{d^2M}{dx^2} - \frac{1}{N}\frac{dN}{dx}\frac{dM}{dx} + \frac{N}{EJ}M = 0. \tag{8}$$

It is difficult to obtain the exact solutions for general cases, but it is possible to solve them for special cases. It is obvious that the exact solutions are dependent on the distribution of flexural stiffness of the bar and axial loads acting on the bar. In this paper, several important cases are discussed.

Case 1. *Expressions of flexural stiffness and axial loads are exponential functions*

$$EJ(x) = \alpha e^{-\beta\frac{x}{l}} \quad N(x) = a e^{-b\frac{x}{l}}. \tag{9}$$

Substituting eqn (9) into eqn (8) we deduce

$$\frac{d^2M}{dx^2} + \frac{b}{l}\frac{dM}{dx} + \frac{a}{\alpha}e^{(\beta-b)\frac{x}{l}} M = 0. \tag{10}$$

Let

$$t = e^{\frac{cx}{2l}}, \quad v = \frac{b}{b-\beta}, \quad M = t^v z, \quad \lambda^2 = \frac{4al^2}{\alpha(\beta-b)^2}, \tag{11}$$

then eqn (10) becomes a Bessel equation of the vth order:

$$\frac{d^2z}{dt^2} + \frac{1}{t}\frac{dz}{dt} + \left(\lambda^2 - \frac{v^2}{t^2}\right)z = 0. \tag{12}$$

The general solution of eqn (10) can be written as

$$M(x) = e^{\frac{cvx}{2l}}[C_1 J_v(\lambda e^{\frac{cx}{2l}}) + C_2 Y_v(\lambda e^{\frac{cx}{2l}})], \tag{13}$$

where v is an integer and $c = \beta - b$.

The boundary conditions of the cantilever are

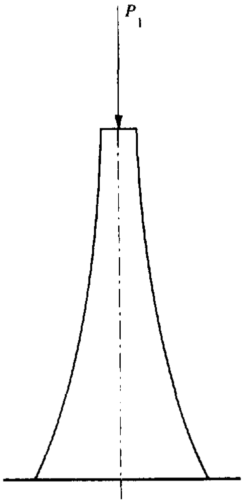

Fig. 3. Special case 1.

$$\frac{dM}{dx} = 0, \quad \text{when } x = 0, \tag{14}$$

$$M = 0, \quad \text{when } x = l. \tag{15}$$

Using these boundary conditions we lead to the following eigenvalue equation:

$$J_{v-1}(\lambda)Y_v(\lambda\theta) = J_v(\lambda\theta)Y_{v-1}(\lambda), \quad v = \text{integer}, \tag{16}$$

in which

$$\theta = k_1 k_2 = e^{\frac{\beta - b}{2}}, \quad k_1 = \sqrt{\frac{J_0}{J_l}}, \quad k_2 = \sqrt{\frac{N_l}{N_0}}. \tag{17}$$

Solving eqn (16) we determine an infinite number of eigenvalues $\lambda_i (i = 1, 2, \ldots)$. Substituting the minimum λ into eqn (11), we derive the critical value of a, and then, by way of eqn (9), we obtain the critical axial loads.

The special cases can be found from the general solution as follows.

(1) When $b = 0$, $\beta \neq 0$, $N(x) = N(l) = P_l$, the general solution becomes that of a non-uniform cantilever subjected to a concentrated axial load at the top of the cantilever (Fig. 3).

In this case, $v = 0$, eqn (12) becomes a Bessel equation of zero order, with eigenvalue equation

$$J_1(\lambda)Y_0(\lambda k_1) = J_0(\lambda k_1)Y_1(\lambda). \tag{18}$$

(2) When $\beta = 0$, $b \neq 0$, $EJ(x) = EJ_0 = \text{constant}$, the general solution becomes that of

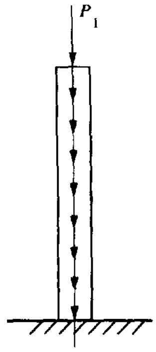

Fig. 4. Special case 2.

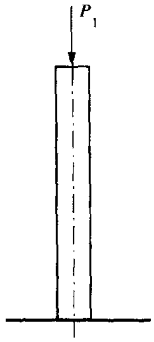

Fig. 5. Special case 3.

a uniform cantilever subjected to concentrated and variably distributed axial loads (Fig. 4).

In the case, $v = 1$, eqn (12) is a Bessel equation of the first order. Its eigenvalue equation is

$$J_0(\lambda)Y_1(\lambda k_2) = J_1(\lambda k_2)Y_0(\lambda). \tag{19}$$

(3) When $b = \beta = 0$, eqn (10) becomes a differential equation with constant coefficients which represents the case of a uniform cantilever subjected to a concentrated load at the top (Fig. 5).

If v is a non-integer, then the eigenvalue equation is

$$J_{v-1}(\lambda)J_{-v}(\lambda\theta) = -J_v(\lambda)J_{-(v-1)}(\lambda\theta). \tag{20}$$

Case 2. *Expressions of flexural stiffness and axial loads are power functions*

$$EJ(x) = \alpha(1+\beta x)^b, \quad N(x) = a(1+\beta x)^c. \tag{21}$$

Substituting eqn (21) into eqn (9) one obtains

$$\frac{d^2M}{dx^2} - \frac{c\beta}{1+\beta x}\frac{dM}{dx} + \frac{a}{\alpha}(1+\beta x)^{c-b}M = 0. \tag{22}$$

Let

$$\zeta = 1+\beta x, \quad M = \zeta^{\frac{1+c}{2}}Z; \tag{23}$$

then eqn (22) becomes

$$\frac{d^2Z}{d\zeta^2}+\frac{1}{\zeta}\frac{dZ}{d\zeta}+\left[n^2\zeta^{c-b}-\frac{(1+c)^2}{4\zeta^2}\right]Z=0. \tag{24}$$

where $n^2 = a/\alpha\beta^2$.

Let

$$t=\frac{n}{k}\zeta^k,\quad k=\frac{c-b+2}{2},\quad r=\frac{1+c}{2k}=\frac{1+c}{c-b+2},$$

then, eqn (24) becomes a Bessel equation of the rth order, namely

$$\frac{d^2Z}{dt^2}+\frac{1}{t}\frac{dZ}{dt}+\left(1-\frac{r^2}{t^2}\right)Z=0. \tag{25}$$

When r is an integer, the general solution of eqn (25) is

$$M(x)=(1+\beta x)^{-\frac{1+c}{2}}\left\{C_1J_r\left[\frac{n}{k}(1+\beta x)^k\right]+C_2Y_r\left[\frac{n}{k}(1+\beta x)^k\right]\right\}. \tag{26}$$

The boundary conditions, eqns (14) and (15), give the eigenvalue equation similar to eqn (16) for r = integer, or similar to eqn (20) for r = non-integer, where

$$\lambda=\frac{n}{k},\quad \theta=(1+\beta l)^k. \tag{27}$$

It is necessary to point out that $r=\infty$ when $b=c+2$. So the general solution found above does not hold in this case. But, it can be seen that eqn (24) becomes an Euler equation. If we let $t=\ln\zeta$, then, eqn (24) is reduced to

$$\frac{d^2Z}{dt^2}+A_1Z=0, \tag{28}$$

in which

$$A_1=n^2-\left(\frac{1+c}{2}\right)^2. \tag{29}$$

When $n<(1+c)/2$, the general solution of eqn (22) becomes

$$M(x)=C_1(1+\beta x)^{-\frac{1+c}{2}+\sqrt{|A_1|}}+C_2(1+\beta x)^{-\frac{1+c}{2}-\sqrt{|A_1|}}, \tag{30}$$

with eigenvalue equation

$$[n^2+2|A_1|-(1+c)\sqrt{|A_1|}]d^{2\sqrt{|A_1|}}=n^2. \tag{31}$$

When $n>(1+c)/2$, the general solution of eqn (22) takes the form

$$M(x)=(1+\beta x)^{-\frac{1+c}{2}}\{C_3\cos[\sqrt{A_1}\ln(1+\beta x)]+C_4\sin[\sqrt{A_1}\ln(1+\beta x)]\}, \tag{32}$$

and the corresponding eigenvalue equation may be written as follows:

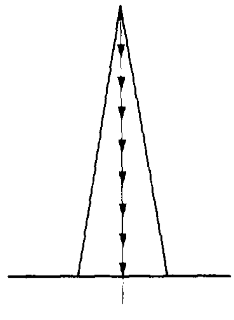

Fig. 6. Cuneiform bar.

$$tg(\sqrt{A_1}\ln d) = \frac{2\sqrt{A_1}}{1+C}, \tag{33}$$

in which

$$d = 1+\beta l. \tag{34}$$

The special cases can be found from the general solution as follows:

(1) When $c = 0$, $b \neq 0$, $N(x) = N(l) = P_l$, the general solution becomes that of the structure shown in Fig. 3.

In this case, $r = 1/(2-b)$. When $b = 2$, then $r = \infty$, and the general solution mentioned above does not hold true; but it is the special example of $b = c+2$. The equation and its solution can be found by the use of eqns (28)–(33), if we let $c = 0$.

(2) When $b = 0$, $c \neq 0$, then $EJ(x) = EJ_0 =$ constant, and the general solution becomes that of the case shown in Fig. 4.

In this case, $r = (c+1)/(c+2)$.

(3) When $c = b = 0$, the case shown in Fig. 5 is obtained.

(4) When $b = -(1/l)$, the general solution becomes that of a cuneiform bar shown in Fig. 6.

STABILITY ANALYSIS OF NON-UNIFORM BARS WITH OTHER SUPPORTS SUBJECTED TO CONCENTRATED AND VARIABLY DISTRIBUTED AXIAL LOADS

In general, C_0 is not equal to zero for non-uniform bars with other than cantilever supports. So, the general solution consists of the complementary solution obtained above and a particular solution. The complementary solution can be written in a unified form as

$$M_0(x) = C_1 T_1(x) + C_2 T_2(x), \tag{35}$$

in which

$$T_1 = t^r J_r(t)$$

$$T_2 = \begin{cases} t^r Y_r(t), & r = \text{integer} \\ t^r J_{-r}(t), & r = \text{non-integer} \end{cases}$$

and the parameters t, r are defined by eqns (12) or (24).

A particular solution for eqn (10) can be found by the use of the complementary solution, eqn (35), as follows:

$$M^*(x) = \begin{cases} \dfrac{C_0 2\pi bl}{C^2} \dfrac{t^{-v-1}}{v!} \sum\limits_{i=0}^{\infty} [J_v Y_{v+i+1} - Y_v J_{v-i-1}] \left(\dfrac{2}{t}\right)^i (v+i)!, & v = \text{integer} \\ -\dfrac{C_0 2\pi bl}{C^2} \dfrac{1}{\sin v\pi \Gamma(v+1)} \sum\limits_{i=0}^{\infty} [J_v J_{-v+i} + J_{-v} J_{v-i}] \left(\dfrac{t}{2}\right)^i \Gamma(v-i), & v = \text{non-integer}. \end{cases} \tag{36}$$

Also, a particular solution for eqn (22) is

$$M^*(x) = -C_0 \frac{(1+\beta x)}{\beta} {}_1F_2\left(1;\ \frac{1}{2k}+1,\ \frac{-c}{2k}+1,\ -\frac{t^2}{4}\right), \tag{37}$$

where ${}_1F_2\,(\cdot)$ is a super geometric series

$$F_2\left(1: \frac{1}{2k}+1, -\frac{c}{2k}+1, -\frac{t^2}{4}\right)$$

$$= \Gamma\left(\frac{1}{2k}+1\right)\Gamma\left(\frac{-c}{2k}+1\right) \sum_{i=0}^{\infty} \frac{F(i+1)}{\Gamma\left(i+\dfrac{1}{2k}+1\right)\Gamma\left(i-\dfrac{c}{2k}+1\right)} \cdot \frac{\left(\dfrac{t^2}{4}\right)^i}{i!}. \tag{38}$$

By now, the solution of eqn (7) may be written as

$$M(x) = C_0 T_0(x) + C_1 T_1(x) + C_2 T_2(x), \tag{39}$$

in which

$$T_0(x) = \frac{M^*(x)}{C_0}.$$

T_0 does not relate to C_0.

Considering the definition of C_0, eqn (6), and the general solution, eqn (39), we derive

$$y'(x) = C_0 \frac{1+T_0'}{N(x)} + C_1 T_1'(x) + C_2 T_2'(x). \tag{40}$$

In order to obtain the eigenvalue equation, it is necessary to consider three boundary conditions. As is well known, the boundary conditions of a pinned–pinned bar with variable cross section are

$$Y''(l) = 0, \quad Y''(0) = 0, \quad Q(l) = Q(0). \tag{41}$$

The substitution of the above conditions into eqns (39) and (40) furnishes three linear equations. Setting the coefficients of C_0, C_1 and C_2 equal to zero we deduce the eigenvalue equation.

Three boundary conditions for a cantilever are also considered; these are

$$Y'(0) = 0, \quad M'(0) = 0, \quad M(l) = 0. \tag{42}$$

The method for a cantilever mentioned in the previous paragraph is simpler than that of this paragraph.

ADDITIONAL METHOD OF SPECIAL CASES

(1) Non-uniform cantilever subjected to a concentrated axial load at the top.

Of course, this problem is only a special case of the previous paragraph, which can be solved by the method mentioned in that paragraph. There is an additional assumption for the problem under consideration.

It is assumed that

$$\frac{1}{EJ(x)} = \alpha\, e^{\beta\frac{x}{l}} + b. \tag{43}$$

There are three parameters in eqn (43), more than in eqn (9). So, this expression is better for describing the distribution of the flexural stiffness of the bar.

If $x = 0, l/2$, and l are selected as control points, then

$$\left.\begin{aligned} b &= \frac{EJ_{l/2}^2 - EJ_0 EJ_l}{EJ_{l/2}[EJ_{l/2}(EJ_0 + EJ_l) - 2EJ_0 EJ_l]} \\ \alpha &= \frac{1}{EJ_0} - b \\ \beta &= \ln\frac{1 - bEJ_l}{\alpha EJ_l} \end{aligned}\right\}. \tag{44}$$

Substituting eqn (43) into eqn (8), we produce

$$\frac{d^2M}{dx^2} + N\alpha\left(e^{\frac{\beta x}{l}} + \frac{b}{\alpha}\right)M = 0. \tag{45}$$

Let us put

$$t = e^{\frac{\beta x}{l}}.$$

Then eqn (45) becomes a Bessel equation

$$\frac{d^2M}{dx^2} - \frac{1}{t}\frac{dM}{dx} + \left(a^2 - \frac{v^2}{t^2}\right)M = 0, \tag{46}$$

in which

$$\left.\begin{aligned} a^2 &= \frac{4\alpha N l^2}{\beta^2}, \quad ,\alpha > 0 \\ v^2 &= \frac{4(-b)Nl^2}{\beta^2}, b < 0 \end{aligned}\right\}. \tag{47}$$

The general solution of eqn (45) is

$$M(x) = \begin{cases} C_1 J_v(a\, e^{\frac{\beta x}{2l}}) + C_2 J_{-v}(a\, e^{\frac{\beta x}{2l}}), & v = \text{non-integer} \\ C_1 J_v(a\, e^{\frac{\beta x}{2l}}) + C_2 Y_v(a\, e^{\frac{\beta x}{2l}}), & v = \text{integer.} \end{cases} \tag{48}$$

Using the boundary condition, eqn (15), we derive the eigenvalue equation

$$J_v(k_3a)[aJ_{-v-1}(a)+vJ_{-v}(a)] = J_{-v}(k_3a)[aJ_{v-1}(a)+vJ_v(a)], \quad v = \text{non-integer} \tag{49}$$

or

$$J_v(k_3a)[aY_{v-1}(a)-vY_v(a)] = Y_v(k_3a)[aJ_{v-1}(a)-vJ_v(a)], \quad v = \text{integer} \tag{50}$$

where

$$k_3 = \sqrt{\frac{1-bEJ_l}{\alpha l}}. \tag{51}$$

(2) Uniform cantilever subjected to a concentrated axial load at the top and variable distributed axial loads (Fig. 4).

This problem is also a special case of the previous paragraph. There is an additional assumption for the problem under consideration.

It is assumed that

$$N(x) = a\,e^{-b\frac{x}{l}} + c. \tag{52}$$

Let $C_0 = 0$, $y' = \varphi$, $M = -EJ\varphi'$, then, eqn (5) becomes

$$EJ\frac{d^2\varphi}{dx^2} + \frac{d(EJ)}{dx}\frac{d\varphi}{dx} + N\varphi = 0, \tag{53}$$

where φ is the angle of cross-section rotation. Substituting eqn (52) into eqn (53) we obtain

$$\frac{d^2\varphi}{dt^2} + \frac{a}{EJ}\left(e^{-b\frac{x}{l}} + \frac{c}{a}\right)\varphi = 0. \tag{54}$$

Letting

$$t = e^{-\frac{bx}{2l}} \tag{55}$$

eqn (54) becomes a Bessel's equation of vth order, i.e.

$$\frac{d\varphi}{dt^2} + \frac{1}{t}\frac{d\varphi}{dt} + \left(\lambda^2 - \frac{v^2}{t^2}\right)\varphi = 0, \tag{56}$$

in which

$$\left.\begin{aligned} \lambda^2 &= \frac{4al^2}{b^2EJ}, \quad && a > 0 \\ v^2 &= \frac{4(-c)l^2}{\beta^2EJ}, \quad && c < 0 \end{aligned}\right\}. \tag{57}$$

The general solution of eqn (54) takes the form:

$$\varphi(x) = C_1J_v(\lambda e^{-\frac{bx}{2l}}) + C_2J_v(\lambda e^{-\frac{bx}{2l}}), \quad v = \text{non-integer} \tag{58}$$

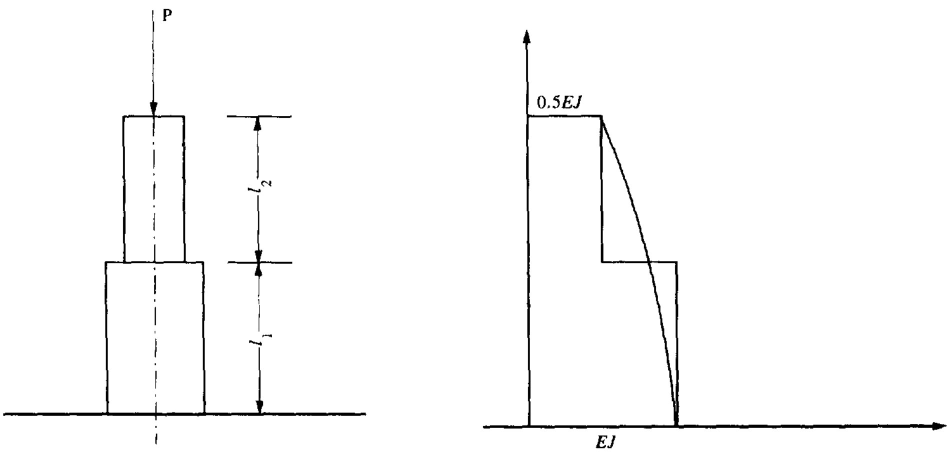

Fig. 7. Bar with stepwise variation in cross section.

$$M(x) = EJ\varphi'(x)$$

$$= EJ\frac{b}{2l}\{C_1[\lambda e^{-\frac{bx}{2l}}J_{r-1}(\lambda e^{-\frac{bx}{2l}})-vJ_r(\lambda e^{-\frac{bx}{2l}})]+C_2[\lambda e^{-\frac{bx}{2l}}J_{-r-1}(\lambda e^{-\frac{bx}{2l}})$$

$$+vJ_{-r}(\lambda e^{-\frac{bx}{2l}})]\}. \quad (59)$$

The eigenvalue equation can be found as follows:

$$J_r(\lambda)[k_4\lambda J_{-r-1}(k_4\lambda)+vJ_{-r}(k_4\lambda)] = J_{-r}(\lambda)[k_4\lambda J_{r-1}(k_4\lambda)-vJ_r(k_4\lambda)] \quad (60)$$

where

$$k_4 = \sqrt{\frac{P_l-c}{a}}. \quad (61)$$

When r = integer, J_{-r} must be changed into Y_r, and the coefficient v of J_{-r} must be shifted to $-v$.

Example

To determine the critical axial force of a bar consisting of two segments and subjected to an end concentrated load shown in Fig. 7, the eigenvalue equation can be found by using the static method as follows

$$tg\alpha_1 l_1 + tg\alpha_2 l_2 = \frac{\alpha_2}{\alpha_1}, \quad (62)$$

in which

$$\alpha_1 = \sqrt{\frac{P}{EJ}} \quad \alpha_2 = \sqrt{\frac{2P}{EJ}} = \sqrt{2}\alpha_1. \quad (63)$$

Substituting eqn (63) into eqn (62) and letting $l_1 = l_2 = l$ one yields

$$tg\alpha_1 l tg\sqrt{2}\alpha_1 l = \sqrt{2}. \quad (64)$$

Solving eqn (64) we derive α_1 and the critical load as follows:

$$\alpha_1 = 0.7190/l$$

$$P_{cr1} = 0.1570EJ/l^2.$$

If the proposed method in this paper is used to solve the above problem, then the step varying distribution of flexural stiffness must be changed to a continuously varying one. If eqn (21) is used at first, the coefficients α, β, b, c are found as

$$b = \tfrac{1}{2}, \quad \alpha = EJ, \quad \beta = -\frac{3}{8l}, \quad c = 0.$$

The order of Bessel function v can be determined as

$$v = \frac{1+c}{c-b+2} = \frac{2}{3}.$$

By solving the eigenvalue equation (20), one obtains

$$P_{cr2} = 0.1576EJ/l^2.$$

The results calculated above show that P_{cr2} obtained by the proposed method is very close to P_{cr1}, that is to the theoretical value achieved by the static method.

It is necessary to point out that the methods mentioned in this paper are exact. Any errors are caused by the difference between the true distribution of flexural stiffness or axial loads and that determined by eqns (9) or (21). So, if a bar has continuously varying distribution or multi-step-varying distribution of flexural stiffness and axial loads, the calculation results will be exact or very close to the exact values.

CONCLUSION

(1) The proposed method and functions describing flexural stiffness and axial loads are not only suitable for stability analysis of bars with varying cross section subjected to variably distributed axial loads, but also for bars with multi-step-varying distribution of flexural stiffness and axial loads.

(2) The special cases, including most of the problems discussed by Timoshenko (1961), Timoshenko (1930), Genik (1950), Jasinsky (1902), Karman and Biot (1940), Vaziri and Xie (1992) and Arbari and Li (1991) and in other previous stability literature, can be found from the general solution proposed in this paper.

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